Table 1 Predicted amplitude ratios for comparison with Fig. 2 from Ref. 5

t(orbits)	$(L/L_o)^{-1}$	$(L/L_o)^{-5/4}$
0	1	1
0.8	2.54	3.2
1.6	5.63	8.7
2.4	13.3	25

tudinal vibration, is $L^{1/2}$. The estimates of Ref. 3 thus imply that the energy of lateral vibration grows during retrieval, whereas the energy of longitudinal vibration shrinks.

Reference 4 treats only lateral vibration by an approximate analysis of a governing linear partial differential equation. This is the piano string equation with constant tension, and with convective temporal derivative due to axial retrieval velocity, but with the assumption that the motion of the line connecting the two attachment points is fixed in a Newtonian reference frame; the terms involving rotation of the "tether frame" are ignored. Ignoring these terms was justified in Ref. 1 by the claim of spectral separation between the vibrational motion and the rotational motion of the system. Reference 1 further argues that approximating the convective derivative by ignoring the term due to retrieval rate is consistent with the assumption of spectral separation since the retrieval rate remains slow. It is this approximation that Kalaycioglu and Misra[4] do not make, and it leads them to the prediction that the amplitude of lateral vibrations grows as $L^{-1/4}$. Further mathematical approximations are made to derive this result, but they appear justified. The assumption that the lateral deformation consists of a single mode, with shape independent of length, is also made.

Finally, one can discuss the results of the published numerical simulations. Figure 7 from Ref. 2, used by Misra in his comment, indeed shows a slow growth of the two generalized coordinates A_1 and A_2 (presumably the amplitudes of two pinned-pinned sinusoidal shapes). This growth is consistent with the prediction that the amplitude grows according to $L^{-1/4}$; this prediction would lead to an increase in amplitude by a factor of 2.2 during a slow retrieval from 100 km to 4 km.

A comparison with the paper of Misra et al.[5] yields a different conclusion. This paper discretizes the lateral tether deflection with pinned-pinned sinusoidal shape functions, but scales the deflection with instantaneous length; out-of-plane deflection is given by

$$u(y,t) = L\sum_{i=1}^{n} A_i(t)\sqrt{2}\,\sin(i\Pi\, y/L) \tag{1}$$

No growth of lateral displacements would thus be indicated by a simulation in which A_i varies as L^{-1}. An $L^{-1/4}$ growth of lateral displacement would be indicated by A_i growing with $L^{-5/4}$. Unfortunately, Ref. 2 shows plots of $A_i(t)$, and the reader must attempt to recognize the difference between L^{-1} and $L^{-5/4}$ trends. I have attempted this, and would summarize this attempt with reference to Fig. 2 from Ref. 5 and Table 1 of this reply. The retrieval rate is relatively simple and slow:

$$\dot{L} = -2\times10^{-4}L \tag{2}$$

where T is the orbital period, taken to be 5400 s, so

$$L/L_0 = \exp\,(-1{\cdot}08\; t/T) \tag{3}$$

It is clear from inspection of Fig. 2 that neither A_1 nor A_2 grows even as quickly as L^{-1}, rather much more slowly. This implies that the lateral displacements, proportional to $L\,A_1$ and $L\,A_2$, shrink.

Conclusions

References 3 and 4 predict that lateral tether vibration amplitudes grow with $L^{-1/4}$ during retrieval. This prediction appears well-matched by a simulation result imprecisely reported in Ref. 2.

Reference 1 predicts that lateral tether vibration amplitudes remain approximately constant during slow retrieval, if energy is conserved.

Reference 5 reports a numerical simulation which predicts that lateral tether vibration amplitudes shrink during slow retrieval, but that this amplitude shrinks slightly more slowly than the length.

It is apparent that there is some disagreement.

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Comment on "Efficacy of the Gibbs-Appell Method for Generating Equations of Motion for Complex Systems"

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Introduction

REFERENCE 1 is the latest in a series of papers by Professor Desloge (see Refs. 2–4) on the Gibbs-Appell method for formulating equations of motion and its relationship to Kane's method. Like the previous papers, the present one contains invalid statements. Although Desloge's claims concerning the independence and relative merits of these two methods have been refuted previously (see, for example, Refs. 5–10), the defects of his latest paper are so serious that it is necessary to set the record straight once more.

In Ref. 1, Desloge applies the Gibbs-Appell method to the formulation of equations of motion of a system consisting of a rigid body and a particle whose motion relative to the body is subject to arbitrary kinematical constraints. He then specializes these equations to those given by Kane and Levinson[11] for a rigid body carrying a light four-bar linkage on whose coupler bar a particle is free to slide. Desloge thereupon claims, without substantiation, that it requires less labor to obtain the equations of motion in this manner than by employing Kane's method as in Ref. 11. This claim has no substance because the

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problem considered by Desloge is not the same one discussed by Kane and Levinson. However, if one applies Kane's method to Desloge's problem, namely the one involving *arbitrary* kinematical constraints, one finds that Kane's method leads to the same equations as those obtained by Desloge using the Gibbs-Appell method, but does so with considerably less labor. Indeed, for *any* problem, Kane's method entails less labor than the Gibbs-Appell method, and the more complex the problem, the greater the savings in labor. Formulating generalized inertia forces by constructing a Gibbs function, a quantity that is of no interest in its own right, and subsequently taking partial derivatives of this function, requires substantially more labor than the simple operations one performs using Kane's method, which are the following: identify partial angular velocities and partial velocities by inspection of angular velocity and velocity expressions, and dot-multiply with inertia torques and accelerations.

Desloge also claims in Ref. 1 that use of Appell's force function to obtain generalized active forces requires less labor than Kane's method. A simple examination of the steps involved in both procedures reveals that the opposite is the case. To obtain generalized active forces with Kane's method, all one has to do is dot-multiply forces with partial velocities and torques with partial angular velocities. In constrast, use of Appell's force function requires one to first dot-multiply forces with accelerations and torques with angular accelerations, tasks significantly more laborious than the dot-multiplications performed in connection with Kane's method. This is so because partial velocities and partial angular velocities are given by simpler expressions than are accelerations and angular accelerations, and, in the case of complex dynamical systems, this disparity is enormous. But the Gibbs-Appell method requires still more work. After the lengthy dot-products have been formed to construct Appell's force function, a quantity that, like the aforementioned Gibbs function, is totally useless in its own right, one must take partial derivatives of this function. Thus, it is clear that the Gibbs-Appell method requires significantly more labor also in this regard than does Kane's method.

The use of Appell's force function suffers from yet another major shortcoming: it can lead to incorrect equations of motion when forces and torques of interest depend functionally on acceleration and/or angular acceleration, as is the case, for example, when Coulomb friction comes into play or certain feedback control laws are employed. For instance, Appell's force function must give rise to wrong equations of motion for a spinning top sliding on a rough horizontal support. Conversely, for whatever forces one encounters, Kane's method always leads to correct equations of motion. Therefore, the Gibbs-Appell method is a special case of Kane's method, not the other way around as Desloge contends.

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Reply by Author to David A. Levinson and Arun K. Banerjee

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THE preceding Comment of Levinson and Banerjee[1] distorts my views,[2–5] hence a synopsis of my position is given before their criticisms are considered.

The standard form of the Gibbs-Appell equations of motion can be written [Ref. 5, Eq. (9)]

$$\frac{\partial}{\partial \ddot{r}_j}\left(\frac{1}{2}\sum_i m_i \ddot{x}_i^2\right) = \sum_i f_i \frac{\partial \dot{x}_i}{\partial \dot{r}_j} \tag{G}$$

If the derivative on the left-hand side of Eq. (G) is carried out and use is made of the identity $\partial \ddot{x}_i/\partial \ddot{r}_j = \partial \dot{x}_i/\partial \dot{r}_j$ [Ref. 5, Eq. (5)], one obtains Kane's equations of motion

$$\sum_i m_i \ddot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{r}_j} = \sum_i f_i \frac{\partial \dot{x}_i}{\partial \dot{r}_j} \tag{K}$$

Equations (G) and (K) are essentially the same basic equation and when applied to the same system lead to the same final result X.

For a given system there are many routes from the general equations of motion (G) to the particular equations of motion X. The force term may be evaluated directly as given, or with the help of the concept of virtual work, or with the use of generalized potentials. The differentiation on the left-hand side of Eq. (G) may be carried out immediately or after the summation has first been simplified or reformulated in any of a variety of ways. Any of these routes from Eq. (G) to X is an application of the Gibbs-Appell method. Advocates of Kane's method are not as liberal. There is an orthodox route from Eq. (K) to X, which they identify as Kane's method. The concept of virtual work is considered to be objectionable[6]; generalized potentials are considered to be useless[1]; and no thought is ever given to reformulating the inertial term to take advantage of the options available in the Gibbs-Appell form of this term.

Since the route from Eq. (G) to Eq. (K) to X is a viable option in the Gibbs-Appell method, it follows that advocates of Kane's method can never claim that their method saves more than the trivial amount of labor required to go from Eq. (G) to Eq. (K). Furthermore, to prove that the labor involved in going from Eq. (K) to X by the orthodox route is less than the labor involved in going from Eq. (G) to X by any

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